The Value of Liquidity

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Abstract
We present a game-theoretic example that helps to illustrate the value of liquidity. These insights are applicable to hedge fund investors since hedge funds have different lock-up and redemption terms. This game also shows the danger of relying on intuition to determine the value of liquidity. We also demonstrate that the value of liquidity is different for different types of investors; the value is less for investors with less ability. Liquidity has become an increasingly important issue in the alternative investments and derivatives space, and this paper provides some quantitative insights on the value of liquidity.

Keywords
Liquidity Risk, Liquidity, Game Theory, Hedge Funds, Lock-up period, Behavioral Finance

Liquidity risk arises from not being able to pull one's money out of an investment instantaneously at fair price. It can have a powerful impact on the portfolio as witnessed by the 1993 Metallgesellschaft debacle (Smithson [1998]). Liquidity is becoming a growing issue in the hedge funds arena where increased regulatory pressure by the SEC has resulted in several hedge funds lengthening their lockup period to two years in order to avoid more scrutiny. Even though liquidity risk is important, very little work has been done to quantitatively understand it in the context of lockup periods. This paper, via an illustrative example, demonstrates the impact of liquidity.

The Game – Balls in a Hat

"Business is a game." — IBM founder Thomas J. Watson

In physics and other physical sciences, it is common to create deliberately simplified "toy" models in the expectation that understanding of the model will lead to intuition about the fuller problem. In this paper we present such a model for the value associated with the lockup period in a hedge fund, in the form of a one person game. We are also inspired by game theory, first invented by the mathematician John von Neumann, which has had many useful applications in economics and finance (Thomas [1986]).

Consider a hat with b black balls and w white balls. At each turn the player chooses whether to draw out a single ball at random, without replacement. The game ends when the player chooses not to remove any further balls or when the hat is empty. The player gains $1 for each white ball drawn, and loses $1 for each black ball drawn. An important feature is that without replacement, the player's draws affect the relative probabilities of subsequent draws.

We consider this game an analogue of investing in a hedge fund. While this analogy might not be readily apparent up front we draw out the comparisons where they arise. This game allows for a rich interweaving of strategies, risk aversion and, most importantly for our present purposes, the cost of liquidity. The qualitative conclusions bear on the real world situation of hedge fund investing.

Numerical Solution

This problem can be solved recursively. In a hat with b black balls and w white balls, suppose that the player chooses a white ball; she is left with $1 plus a hat with b black balls and $w-1$ white balls. If she picks a black ball she is left with −$1 plus a hat with $b-1$ black balls and w white balls. The probability of realizing either outcome is given by the ratios of the numbers of each color of ball to the total number. The decision of whether or not to play is determined by whether the expected winnings under the two scenarios are positive. Defining the value of a hat by $E(b,w)$, the consideration above can be expressed mathematically as

$$E(b,w) = \max\left[\frac{w}{b+w}(1+E(b,w-1)) + \frac{b}{b+w}(-1+E(b-1,w)), 0\right].$$

To make this complete, we specify the boundary conditions that if we have only white balls then the value is the number of white balls and if we have only black balls then the value is zero. These conditions are expressed $E(0,w) = w$ and $E(b,0) = 0$.

Exhibit 1. The value for various choices of *w* and *b*

white\black	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	1	0.50	0	0	0	0	0	0	0
2	2	1.33	0.67	0.20	0	0	0	0	0
3	3	2.25	1.50	0.85	0.34	0	0	0	0
4	4	3.20	2.40	1.66	1.00	0.44	0.07	0	0
5	5	4.17	3.33	2.54	1.79	1.12	0.55	0.15	0
6	6	5.14	4.29	3.45	2.66	1.91	1.23	0.66	0.23
7	7	6.13	5.25	4.39	3.56	2.76	2.01	1.34	0.75
8	8	7.11	6.22	5.35	4.49	3.66	2.86	2.11	1.43

As mentioned, the decision of whether or not to play is determined solely by whether the expected winnings under the two scenarios is positive; we call that strategy "optimal". Later we will consider more "risk-averse" strategies of play which sacrifice some expected earnings for lesser uncertainty in the outcome; these can be formulated as a variant on the formula above.

Each cell in Exhibit 1 represents the solution to the recursion relation and is the expected value of employing the optimal strategy to the corresponding. The matrix is a solution template and indicates how to effectively play the game; the player picks a ball if the value is positive, and does not pick a ball if the value is zero.

As an illustrative example, consider the hat containing one white ball and one black ball. If a white ball is chosen, the player would elect to stop and gain \$1. If a black ball is chosen, she would pick again and be assured of getting a white ball, since it would be the last ball remaining, for a net gain of zero. Consequently, the expected value is $0.5(1) + 0.5(0) = 0.5$ which is the displayed value for that cell.

Understanding the Solution

The results of this game help illustrate the value of the player's right to stop. Consider the hat of size six black balls and four white balls (lightly highlighted in the matrix above). Intuitively, one might think that it is not worth playing since there are more black balls than white balls. Surprisingly, the expected value of this game is positive; equal to 1/15. Thus, it makes sense to play this hat as well. The reason is that the option of being able to stop at any time overcomes the imbalance of black balls to white.

This result extends to where there are more white balls than black balls. Consider the case of six white balls and four black balls (the cell with darker highlighting in Exhibit 1.) The naive player might assume that the value is \$2 but in fact there is substantially more expected payoff than that. There is an excess of \$0.66 in this situation, arising from the fact that even if the game strays to a situation where $b > w$, by choosing when to exit the player has a substantial probability of recouping losses. Again we see that the option to stop playing acts as a buffer against experiencing substantial losses.

A later section of this paper explores the behavior of this game in the asymptotic case (*i.e.*, as b and w go to infinity). We also examine other strategies since behavior finance has demonstrated that investors are not always rational (Nofsinger [2005])) and have different risk appetites. But first we draw out the analogy to hedge funds and liquidity more explicitly.

Analogy to Hedge Funds

The connection to hedge fund investing is the value which arises from being able to time one's exit. By acting optimally the player can participate in all of the upside of drawing more than the expected number of white balls while not fully participating in the downside of drawing more than the expected number of black balls. Similarly, a judicious hedge fund investor can pull her money in an optimal manner so as to reap profits even when the fundamentals of the investment do not look promising.

Even though good liquidity is regarded as being a desirable property, it is not well understood quantitatively. This is demonstrated by the fact that given the return series of two hedge funds, most portfolio managers conduct statistical analysis to compare the two (mean, standard deviation, skew, kurtosis, Sharpe, Omega, etc.) while ignoring the difference in liquidity. Many firms do not have adequate statistics to measure liquidity risk (Bhaduri and Kaneshige [2005]). The *Balls in the Hat* game shows that it can be dangerous to rely on intuition regarding liquidity matters.

Approximate Solution

When the number of balls is very large we can make analytical headway in analyzing the value of the game. In that limit, the recursion relation becomes a differential equation which has an explicit solution of

$$E(b, w) \approx \begin{cases} w - b + A\sqrt{2\pi N}\exp(2t^2)(\mathrm{erf}(\sqrt{2}t) + 1) & \text{for } t < t_c \\ 0 & \text{for } t \geq t_c \end{cases}$$

Exhibit 2. The asymptotic approximation

white\black	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	1.19	0.52	0.08	0	0	0	0	0	0
2	2.22	1.42	0.73	0.22	0	0	0	0	0
3	3.24	2.38	1.59	0.90	0.35	0.03	0	0	0
4	4.25	3.36	2.52	1.74	1.04	0.47	0.09	0	0
5	5.25	4.35	3.47	2.64	1.86	1.16	0.58	0.16	0
6	6.26	5.34	4.44	3.58	2.75	1.97	1.27	0.68	0.23
7	7.26	6.33	5.42	4.53	3.67	2.85	2.08	1.37	0.77
8	8.27	7.33	6.41	5.50	4.62	3.76	2.94	2.17	1.47

where erf is the error function and we have defined $N = b + w$, $t = \sqrt{N}(b/N - 1/2)$ and $A = 0.147$, $t_c = 0.42$. We will demonstrate this in a more technical companion document.

The approximation works well as the number of balls gets large and for values of b and w close to the diagonal. (There is in fact a separate approximation optimized for b much smaller than w, but that is not as interesting a situation.) An important observation about the approximation is there is a very large domain of $b > w$ which supports a nonzero value. In fact, this domain increases with the number of balls in a square root manner.

Further support for the correctness of the approximation is provided in the chart (Exhibit 3) where we show the value for a hat with 100 balls in total as a function of the number of black balls. Obviously the number of white balls is 100 minus the number of black balls. The points are the exact values from the recursion relation while the smooth curve is from the approximation. Clearly the approximation works very well in this situation.

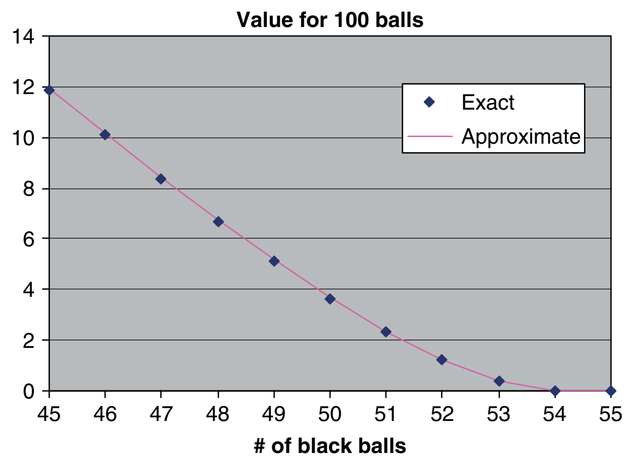

Exhibit 3. The value of a hat with 100 balls as a function of the number of black balls.

Strategies

We have derived results for the case where the player elects to play as long as the expected value of playing is positive. We call this a "strategy" and make this notion precise by defining a function of b and w which indicates whether to keep playing or not. We call this function $s(b, w)$ and it equals one if the player continues according to that strategy and zero if the player stops. The strategy we have enumerated before is unity if the first argument of the max function is positive and zero if it is negative. (In fact we could imagine an even richer set of strategies which incorporate information about previous draws and not just the current configuration, but we forego overburdening the discussion.)

There are other strategies one can envisage. We enumerate three naive strategies

1. the player chooses never to play; $s(b, w) = 0$.
2. the player chooses always to play; $s(b, w) = 1$.
3. the player chooses to play as long as w > b; $s(b, w) = 1$ for $w > b$; 0 otherwise.

The third strategy is clearly preferable to the other two since it is never less than either of them in all situations. The other interesting aspect of these three strategies is that there is nothing random about the outcome. Under these strategies the player will finish with exactly 0, $w - b$ and $\max(w - b, 0)$, respectively.

The solutions of all the strategies we have outlined are governed by the recursion relation

$$E(b, w) = \begin{cases} \frac{w}{b+w}(1 + E(b, w-1)) + \frac{b}{b+w}(-1 + E(b-1, w)) & s(b, w) = 1 \\ 0 & s(b, w) = 0. \end{cases}$$

The strategy of playing as long as the expected payoff is positive can be called "optimal" since it generates the largest expected payoff. However it is relatively risky. For example, with $b = 6$ and $w = 4$, this strategy would argue for continuing to play. However continuing to play also lends itself to the possibility of losing money, and depending on the player's risk appetite, the reward might not be enticing enough for her to

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play. Depending on their risk appetite, a player might elect to have a strategy intermediate between three and the optimal strategy. Such a player would still realize a value from their ability to act in a perfectly liquid manner but would temper their expected earnings to lessen their riskiness.

We will explore in greater detail the competition among liquidity, expected reward and riskiness in a companion document.

Modified game – Balls in the Hat with a Liquidity Constraint

In this game, the modification is that once a player chooses to stop, she must still pick one more ball. The exception to this is if the player chooses not to pick any balls at all since that is analogous to not investing. This modification represents a lockup period in this game. The original game helped to demonstrate the value of liquidity by showing how the power to pull out of the game at any time can help make seemingly unattractive investments profitable. We now explore the effect of injecting a liquidity constraint to the value of the game, thereby limiting the power of pulling out optimally.

Again for each hat we must choose between playing and stopping. However there is now a nonzero value associated with stopping which is the expected return from one additional draw. Let us consider the hat consisting of one ball of each color. In this case, if the player decides to play she might start by picking a white ball in which case she will decide to quit but be forced to play one more time, thereby getting the black ball. Alternately she might start by picking the black ball and after would get the white ball. In either scenario her net winnings are zero and that is the value of the game. This compares to the value of $0.50 in the unconstrained game.

In general, the value of this game is less. We have the following recursion relation for the value, which we now call F (the strategy is still to maximize expected payoff.)

$$F(b,w) = \max\left[\frac{w}{b+w}(1+F(b,w-1)) + \frac{b}{b+w}(-1+F(b-1,w)),\ \frac{w-b}{b+w}\right]$$

The boundary conditions are $F(0,w) = w$ and $F(b,0) = -1$. (The reason for the -1 value is that for a hat with all blacks the best strategy is to immediately announce an intention to withdraw and then lose one dollar while waiting through the lock up period.) In general, there is no guarantee that $F(b, w)$ is positive. The values of the liquidity modified game using the strategy of playing if there is positive expected value are listed in Exhibit 4.

Comparing to Exhibit 1, the examples of the (6,4) hat and the (4,6) hat have changed substantially. Loss of liquidity has completed removed any positive expected value from the first hat and has essentially halved the excess value (relative to the strategy three value of $2) from the second hat.

There is now an important distinction between a player who is already in the game and one who is not. The first player has the option not to start (analogous to choosing not to invest.) From their perspective the table above should have negative values replaced by zeros. Such a player contemplating the (6,4) hat for instance will simply choose not to play, so from their perspective the value of that hat is zero. However the value from the perspective of a player already in the game is negative. We conclude that locking up qualitatively changes the situation. An investor already locked in should assess the investment differently from an investor who is contemplating joining. While this conclusion arises from this arguably contrived game, it is also true for the real world situation of hedge fund investors.

We can extend the game by forcing the player to make n selections after stating an intent to stop. In Exhibit 5 we show the two step delayed game. Now the second argument of the max function in the recursion relation is replaced by the expected payoff of two additional draws. The value is further suppressed relative to the unconstrained game.

The expected value for a player already in the game converges to $w - b$ as the delay n gets large, which is equivalent to strategy two enumerated above. The difference between a player already in the game and a player contemplating entering the game is that the second player would only enter if the value is positive, which is the payoff of strategy three. Therefore the impact of liquidity for large n is to force all players to follow strategy three.

Of course a player who was following strategy three anyway (either because they are not particularly sophisticated or because they are

Exhibit 4. The value of the game with a liquidity constraint

white\black	0	1	2	3	4	5	6	7	8
0	0	−1	−1	−1	−1	−1	−1	−1	−1
1	1	0	−0.33	−0.50	−0.60	−0.67	−0.71	−0.75	−0.78
2	2	1	0.33	−0.20	−0.33	−0.43	−0.50	−0.56	−0.60
3	3	2	1.20	0.50	0.00	−0.25	−0.33	−0.40	−0.45
4	4	3	2.13	1.34	0.67	0.15	−0.20	−0.27	−0.33
5	5	4	3.10	2.25	1.48	0.82	0.26	−0.13	−0.23
6	6	5	4.07	3.19	2.37	1.61	0.94	0.37	−0.03
7	7	6	5.06	4.15	3.29	2.48	1.73	1.05	0.48
8	8	7	6.04	5.12	4.23	3.38	2.58	1.83	1.15

Exhibit 5. The value of the game with a two-step liquidity constraint

white\black	0	1	2	3	4	5	6	7	8
0	0	−1	−2	−2	−2	−2	−2	−2	−2
1	1	0	−0.67	−1.00	−1.20	−1.33	−1.43	−1.50	−1.56
2	2	1	0.17	−0.40	−0.67	−0.86	−1.00	−1.11	−1.20
3	3	2	1.10	0.35	−0.23	−0.50	−0.67	−0.80	−0.91
4	4	3	2.07	1.23	0.50	−0.06	−0.40	−0.55	−0.67
5	5	4	3.05	2.16	1.35	0.65	0.08	−0.33	−0.46
6	6	5	4.04	3.12	2.26	1.47	0.77	0.19	−0.23
7	7	6	5.03	4.09	3.20	2.36	0.00	0.88	0.30
8	8	7	6.02	5.07	4.16	3.28	2.45	1.68	0.99

extremely risk averse) will not be affected by the imposition of the liquidity constraint. This implies that the cost of liquidity depends on the nature of the investor. Sophisticated investors with a greater appetite for risk are prevented from acting in a beneficial manner and experience a large cost of liquidity. Average investors (those following strategy three) would not do any better with their money anyway and are not strongly affected. Poor investors who might otherwise realize less than strategy three may in fact benefit from the imposition of the lockout. From this perspective the effect of the liquidity can be understood as lessening the comparative advantage of being a savvy investor.

Possible Extensions of the Analogy

There are a number of ways in which this game fails to capture essential elements of hedge fund investing. One is that when deciding whether to continue playing, the player knows the distribution of outcomes associated with that choice. That is not the case for the hedge fund investor who needs to infer that distribution either from historical performance or their own idiosyncratic view of the ongoing viability of the hedge fund strategy. This could be addressed by starting with a hat where only a prior distribution of the numbers of balls is known, as opposed to explicitly knowing their starting values. In that event the player would be continually inferring the distribution of the balls from the draws and using that inference as part of the decision making process. This would almost certainly affect the value of the lockup period in an interesting manner.

Another difference is that we have assumed that once a player decides to exit, she would never choose to reenter the game. However if we imagine that draws continue to be made (but without the player experiencing either the benefit or penalty) there could come a time when the player might choose to reenter the game due to an attractive change in the composition of the hat. This would mimic the effect of a hedge fund being unattractive at one time but attractive at a later time as market conditions change. Playing in this manner would change the payoff of the game and would also affect the value of the lockup since the lockup would potentially be experienced on multiple occasions.

Conclusion

The value of liquidity is particularly important in the alternative investments arena. Hedge funds, private equity, and real estate are all usually more illiquid than traditional investments. The *Balls in the Hat* game demonstrates that the value of liquidity can sometimes overcome seemingly bad odds to make a difference of whether or not one should invest. An investor not yet invested in a fund has a fundamentally different view than an investor already in the fund and subject to the lockout. Different investors will be following different strategies, some of them very risk averse (or simply not sophisticated) and others who are not afraid of risk and know how to optimize their expected returns. We observe that the value of liquidity is less for the first type of investor and greater for the second. This is based on the fact that the first class of investors will not be able to take advantage of more favorable liquidity since they are more likely to reinvest the money less lucratively. In addition, we have shown that it is dangerous to rely on intuition alone in trying to determine the value of liquidity, which dovetails well with behavioral finance.

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